Normal Momentum Transfer on Ideal Crystalline Surfaces

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IN a previous Note[1] the authors reported on a molecular collisional analysis of tangential momentum transport from a flowing gas to a solid surface. The surface was taken to be macroscopically smooth and flat, i.e., monocrystalline on a molecular scale, so that comparison with experimental data on tangential momentum transfer rates could be used to estimate the importance of roughness of real surfaces. The gas-surface system was represented by a simple molecular model, and calculations were made of the tangential momentum transfer coefficient σ_t as a function of the dimensionless gas velocity and the length parameter of the molecular interaction potential. The trends in the calculated results for σ_t generally agreed with observation, but the calculated magnitudes were somewhat lower than typical measured values, as would be expected if the effect of surface roughness (polycrystallinity) were significant.

The purpose of this Note is to report values of normal momentum transfer coefficients calculated for the same gas-surface model. In this case the model is clearly of more questionable validity, for reasons to be described below. The results should therefore be viewed as tentative until they can be compared with experiment or until studies with a more accurate surface model have been made.

Consider the free-molecule flow of a rarefied gas along a stationary infinite flat plate whose surface is the plane $y = 0$. The gas velocity $\mathbf{u} = \hat{\mathbf{x}}u(y,t)$ is in the $\hat{\mathbf{x}}$ direction parallel to the plate and is steady and uniform far from the plate; that is, $\lim_{y\to\infty} u(y,t) = U$, a constant. If we let $f(\mathbf{c},y,t)$ be the distribution function for molecular velocities $\mathbf{c}$, the interfacial momentum flux tensor is defined by

$$\mathbf{P}_s = \int m(\mathbf{c} - \mathbf{u}_s)(\mathbf{c} - \mathbf{u}_s)f_s d\mathbf{c} \tag{1}$$

where m is the molecular mass, f_s is the surface function $f(\mathbf{c},0,t)$, $\mathbf{u}_s$ is the gas velocity at the surface given by

$$\mathbf{u}_s = n_s^{-1}\int \mathbf{c} f_s d\mathbf{c}$$

and $n_s = \int f_s d\mathbf{c}$ is the gas density at the surface. The normal momentum accommodation coefficient σ_n is defined as the fraction of the incident flux of normal momentum that is transferred to the surface by collisions

$$\sigma_n = (p_{yy})_s/(p_{yy})_s{}^i \equiv [(p_{yy})_s{}^i - (p_{yy})_s{}^r]/(p_{yy})_s{}^i \tag{2}$$

where $(p_{yy})_s{}^i$ is the normal momentum flux incident on the surface and $(p_{yy})_s{}^r$ is the reflected flux.

In order to evaluate σ_n from Eqs. (1) and (2) we need to know the surface function f_s. For the conditions on the flow just described, the Liouville theorem requires that f_s satisfy

$$f(\mathbf{c}) = \begin{cases} f^{[0]}(\mathbf{c}) & \text{for } c_y < 0 \\ f^{[0]}(\mathbf{c}') & \text{for } c_y > 0 \end{cases} \tag{3}$$

where $\mathbf{c}'$ is the precollisional velocity of a molecule that subsequently leaves the surface with velocity $\mathbf{c}$, and $f^{[0]}(\mathbf{c})$ is the Maxwellian distribution for velocities in the incident stream with uniform density n_0, temperature T_0, and velocity $\mathbf{u}_0 = \hat{\mathbf{x}}U$. The precollisional velocity $\mathbf{c}'$ is related to the postcollisional velocity $\mathbf{c}$ by the dynamics of the collisional process. For mathematical convenience we choose to consider the collision in terms of the postcollisional velocity $\mathbf{c}$, and thus $\mathbf{c}' = \mathbf{c}'(\mathbf{c})$.

The determination of the surface collision equations which give $\mathbf{c}'$ as a function of $\mathbf{c}$ depend on the molecular model used to represent the gas-surface interaction. The model used for the calculations of σ_t in Ref. 1 is defined by the following assumptions: a) the surface atoms can be assumed to be immobile in calculating the collisional transport of momentum; b) the surface may be assumed to be two-dimensional in the sense that the gas-surface potential depends on only two spatial coordinates, x and y; and c) the dependence of the gas-surface potential is assumed periodic in the surface coordinate x, corresponding to the regular placement of the surface atoms in the crystal lattice.

As discussed in Ref. 1, the most limiting of these three assumptions is the first. Although tangential momentum transfer rates are very insensitive to the amount of surface atom motion allowed in the model, such is not the case for normal momentum transfer. The difference in the two cases is demonstrated by the work of McClure,[2] who found for a typical gas-surface system and a gas temperature of 300°K that an increase in the mass of the surface atoms by a factor of 3 produces a 0.1% change in the tangential velocity of the emerging stream, while the normal velocity of the stream increases by 6.9%. Although an estimate from these results of the error incurred in assuming infinitely massive surface atoms is not possible, the sense of the error is known. Since the effect of assuming immobile surface atoms is to increase the normal momentum of the reflected stream, the results obtained here for the normal momentum accommodation coefficient should be low.

With the above assumptions about the model, the results of Ref. 1 may be applied to Eq. (1) to derive

$$(p_{yy})_s{}^i = \frac{2n_0kT_0}{\pi L}\int_{-L/2}^{L/2}\int_0^{\pi}\int_0^{\infty} v^3\cos^2\theta e^{-(v^2+2vV\cos\theta+V^2)}\, dv d\theta dx \tag{4}$$

and

$$(p_{yy})_s{}^r = \frac{2n_0kT_0}{\pi L}\int_{-L/2}^{L/2}\int_0^{\pi}\int_0^{\infty} v^3\cos^2\theta e^{-(v^2+2vV\cos\theta'+V^2)}\, dv d\theta dx \tag{5}$$

where θ and $\theta' = \theta'(x,\theta)$ are the cylindrical polar angles of the velocity vectors $\mathbf{c}$ and $\mathbf{c}'$, V is the dimensionless velocity $(m/2kT_0)^{1/2}U$, and L is the period of the surface potential. Equation (4) may be integrated analytically to give

$$(p_{yy})_s{}^i = n_0kT_0(V^2 + \tfrac{1}{2}) \tag{6}$$

Unfortunately, an analytic solution of the integrals for $(p_{yy})_s{}^r$ in Eq. (5) is not feasible. However, the equation may be put in a form more convenient for numerical evaluation by noting that, since θ' is not a function of v, the v integration in Eq. (5) may be carried out first. This leads to

$$(p_{yy})_s{}^r = \frac{n_0kT_0}{\pi L}e^{-V^2}\int_{-L/2}^{L/2}\int_0^{\pi}\cos^2\theta\left[B^2 + 1 - \pi B^{1/2}\left(B^2 + \frac{3}{2}\right)e^{B^2}\operatorname{erfc}B\right]d\theta dx \tag{7}$$

where $B = V\cos\theta'(\theta,x)$.

The results obtained thus far are valid for any interaction potential between a gas atom and a surface atom. The evaluation of the expression for $(p_{yy})_s{}^r$ in Eq. (7) will be carried out here for the case of impulsive interactions. The resulting model for the gas-surface interaction is described in Ref. 1. Briefly, the gas molecules are rigid spheres of diameter d_g and the surface is a two dimensional square array of atoms of the diameter d_s separated by a distance L. Since the

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Table 1 Calculated values of σ_n

l	V	σ_n[a]
0.25	0.07	$<10^{-3}$
	0.10	0.001
	0.20	0.003
	0.50	0.015
	1.0	0.038
0.50	0.07	0.002
	0.10	0.003
	0.20	0.013
	0.50	0.060
	1.0	0.139
0.75	0.07	0.003
	0.10	0.007
	0.20	0.025
	0.50	0.117
	1.0	0.253
1.0	0.07	0.005
	0.10	0.010
	0.20	0.038
	0.50	0.171
	1.0	0.353
1.2	0.07	0.006
	0.10	0.012
	0.20	0.045
	0.50	0.203
	1.0	0.411
$(2)^{1/2}$	0.07	0.007
	0.10	0.013
	0.20	0.049
	0.50	0.223
	1.0	0.450

[a] For $V < 0.01$ the calculated value of σ_n was $<10^{-3}$ for all l.

surface is assumed uniform in the z direction, it takes the form of an array of rigid cylinders of diameter d_s with their axes lying parallel to the z axis a distance L apart. Thus, the region unavailable to the centers of the gas molecules is bound by a series of segments of infinite parallel cylinders of diameter $d = \frac{1}{2}(d_g + d_s)$. The plane of the "macroscopic" surface $y = 0$ is that tangent to the cylindrical segments, so the centers of the surface atoms lie in the plane $y = -d$. Because the collisions are impulsive and the surface is rigid, the collision function $\theta'(x,\theta)$ can be determined analytically from the geometry of the collision trajectory for given values of the size ratio $l = L/d$.

Values of σ_n were calculated by numerical integration of Eq. (7) on an IBM 7040/7094 computer. The results are given in Table 1 for thirty pairs of values for l and V. In addition to the results given in Table 1, calculations of σ_n were also done for low velocities, $V \leq 0.01$. The values found, however, were all below 10^{-3}, so that the surface model used here indicates that essentially no normal momentum is transferred from the gas to the surface when the dimensionless gas velocity is below 0.01. In fact, inspection of Table 1 shows that normal momentum transport is quite low, even for microscopically rough surfaces (l large), for gas velocities below $V = 0.5$. However, for high gas velocities, this is not the case. A value of $V = 1$ corresponds to a gas velocity of about 1000 fps at 300°K. At this high gas velocity there is a significant amount of normal momentum transfer; $\sigma_n = 0.25 - 0.41$ in the range of $l = 0.75 - 1.2$ typical of most surfaces.

Care should be taken in giving too much quantitative significance to the above results. In contrast with the case of tangential momentum transport, the applicability of an infinitely massive, monocrystalline surface model to normal momentum transport is not clear. As discussed previously, however, the effect of assuming immobile surface atoms is to lower σ_n. Similarly, the assumptions of an ideal two-dimensional crystalline surface and of impulsive interactions also decrease σ_n. These factors strongly suggest that the calculated values of σ_n reported here represent a lower bound. Consequently, the results of this study imply that, for high-speed gas flows, a significant fraction of the incident normal momentum will be transferred to any solid surfaces as a result of gas-surface collisions.

References

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Laminar Boundary Layers with Large Wall Heating and Flow Acceleration

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I. Introduction

THIS Note is concerned with evaluating the simultaneous effects of wall heating and acceleration on the structure of laminar boundary layers. Earlier similarity solutions for perfect gases with viscosity proportional to temperature and Prandtl number of unity by Cohen and Reshotko[1] indicated a beneficial increase in the heat-transfer parameter G_w', to which the heat flux is related, as the heating parameter g_w, the ratio of wall to total enthalpy, was increased when flow acceleration occurred. Their calculations were carried out for values of g_w and $\bar{\beta}$, a parameter descriptive of the free-stream velocity variation, up to 2. A range of g_w and $\bar{\beta}$ up to 5 is considered in this investigation so that predictions will be available for larger wall heating (e.g., in channel flows in nuclear rockets and in resistojets) and for larger freestream velocity variations (e.g., see Ref. 2 for the magnitude of $\bar{\beta}$ found in internal flows). Investigations of the effect of relatively large values of the heating parameter have been carried out for laminar boundary layers, but they apply to flows with no freestream velocity variation ($\bar{\beta} = 0$) for g_w from 1 to 10 (Ref. 3), for g_w from 1 to 6 in which compressibility effects are included,[4] or at a stagnation point with the effect of mass transfer included ($\bar{\beta} = \frac{1}{2}$, axisymmetric body, $\bar{\beta} = 1$, two-dimensional body) for g_w from 1 to infinity.[5]

II. Analysis

The coupled, transformed, laminar boundary-layer equations that are solved numerically are as follows:

$$f''' + ff'' + \bar{\beta}[G(1 - g_w) + g_w - (f')^2] = 0 \quad (1)$$

$$G'' + fG' = 0 \quad (2)$$

The reader is referred to Ref. 6 for a description of the various quantities in Eqs. (1) and (2), and other quantities that appear herein. The application of the solutions to actual flows with variable $\bar{\beta}$ and more realistic property variation is discussed later.

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